Journal of Statistical Physics, Vol. 60, Nos. 5/6, 1990

The Limit-Periodic Finite-Difference Operator on $l^2(\mathbb{Z})$ Associated with Iterations of Quadratic Polynomials

M. Sodin[1] **and P. Yuditski**[2]

Received March 20, 1990

The limit-periodic discrete operator of the Schrödinger type on the axis $\mathbb{Z}$ associated with iterations of quadratic polynomials is investigated. It is proved that this operator has a singularly continuous simple spectrum. Connections with the Sinai–Bowen–Ruelle measure and the conformal map onto a special comblike domain are established.

KEY WORDS: Limit periodicity; Jacobi matrices; iterations of polynomials; Sinai–Bowen–Ruelle measure; escape rate; comblike domains; conformal map.

1. INTRODUCTION

Spectral properties of limit-periodic (LP) Jacobi matrices associated with iterations of polynomials with a real Julia set have been considered.(1) More exactly, what was studied were examples of bounded LP second-order finite-difference operators H_+ on the "semiaxis" $\mathbb{Z}_+$ having a continuous singular spectrum. The spectral measure μ of the operator H_+ is the balanced measure(5) of the iterated polynomial $T(x) = x^2 - \lambda$, $\lambda > 2$. The support of μ coincides with the Julia set J of T having a zero linear measure.(5)

The LP property allows a natural extension of the operator H_+ to the whole "axis" $\mathbb{Z}$. We will study spectral properties of such an extension H. (This question arose during a discussion of papers refs. 1 at the seminar of V. A. Marchenko.) The operator H is an orthogonal sum $H = H_+ \oplus H_-$, which is why we need to analyze only the operator H_- acting in $l^2(\mathbb{Z}_+)$.

[1] Institute for Low Temperature Physics and Engineering, Kharkov, USSR.
[2] Institute for Mechanization and Electrification of Agriculture, Kharkov, USSR.

The main result is the following. The spectral measure ν of the operator H_- is the Sinai–Bowen–Ruelle (SBR) measure[2] of the polynomial T. In other words, it is an "eigenmeasure" of the operator $\mathbb{B}^*$ adjoint to the Ruelle–Perron–Frobenius (RPF)-type operator,

$$(\mathbb{B}g)(x) = \sum_{T(y)=x} \frac{g(y)}{|T'(y)|^2}$$

which acts in $C(J)$. It is known[2] that ν is a purely continuous singular measure supported also on J, and ν is mutually singular with respect to μ. Thus, the spectrum of the extended operator H is continuous, singular, and simple.

There is an interesting and useful connection between a spectrum of differential or difference Schrödinger operators with periodic coefficients and conformal maps of the upper half-plane $\mathbb{C}_+$ onto special comblike domains.[6,7] Recently such a connection was established for some class of operators with LP coefficients.[8] The operator H does not belong to this class, but nevertheless such a connection exists for it as well. In Section 2 we will prove that if $B(z)$ is the Böttcher function of the polynomial $T(x)$ (see refs. 3–5), then the function $\theta(z) = i \log B(z)$ maps $\mathbb{C}_+$ onto the special comblike domain, and the Julia set J is a preimage of the base of this comblike domain. (A similar statement was formulated in ref. 14 in an implicit form.)

In Section 3 we recall some facts on the operator H_+. In Section 4 we prove the main result and in Section 5 we discuss other properties of the operator H. Section 6 is devoted to properties of the SBR measure of T and of the operator $\mathbb{B}$. We find the explicit expression for the Stieltjes transform of the SBR measure (i.e., for the resolvent function of the operator H_-) and for the "principal eigenfunction" $h(x)$ of the operator $\mathbb{B}$.

2. ITERATIONS OF POLYNOMIALS AND A CONFORMAL MAP ONTO A SPECIAL COMBLIKE DOMAIN

Let T_n be the nth iteration of $T(x) = x^2 - \lambda$, $\lambda > 2$, and $\mathcal{D}_T(\infty) = \{z \in \bar{\mathbb{C}}: T_n(z) \to \infty, n \to \infty\}$ be a basin of attraction to infinity. Then $J = \bar{\mathbb{C}} \backslash \mathcal{D}_T(\infty)$ is the Julia set of T. It is a nowhere dense, real, compact set of zero linear measure.[5] In our case the polynomial T is expanding on J, i.e., $\inf\{|T_n'(x)|: x \in J\} \geqslant ac^n$, $c > 1$, $a > 0$.

Let μ be the balanced measure[1,3–5] of T. This means that for every $f \in C(J)$,

$$\int f\, d\mu = \tfrac{1}{2} \int [f((x+\lambda)^{1/2}) + f(-(x+\lambda)^{1/2})]\, d\mu(x) \tag{2.1}$$

It is known[5] that such a measure always exists, is unique, and coincides with the equilibrium measure of J.

Let Π be the semistrip $\Pi = \{z = x + iy: 0 < x < 2\pi,\ 0 < y < \infty\}$ and let $a > 0$; then

$$\Omega(a) = \Pi \setminus \bigcup_{n=0}^{\infty} \bigcup_{q=0}^{2^n - 1} \left\{ z = x + iy: x = \frac{(2q+1)}{2^n},\ 0 \leqslant y \frac{a}{2^n} \right\}$$

is the comblike domain (see Fig. 1). We denote by $\theta(z)$ the conformal map of the upper half-plane onto $\Omega(a)$ with the following boundary correspondence: $\theta(\infty) = \infty$, $\theta(-\xi) = 0$, $\theta(\xi) = 2\pi$, where $2 < \xi < \infty$. Later we will link the parameters ξ and a. Such a map does exist. It is unique, and by virtue of the symmetry of the domain $\Omega(a)$, we get $\theta(0) = \pi + ia$.

Let

$$B(z) = \lim_{n \to \infty} [T_n(z)]^{1/2^n}$$

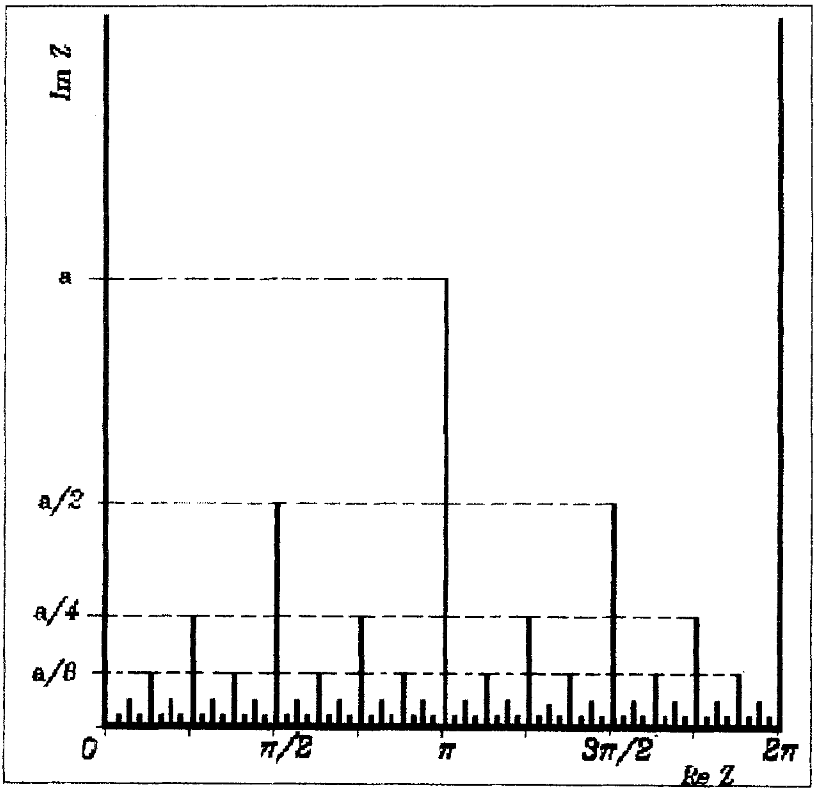

Fig. 1. The comblike domain $\Omega(a)$.

be the Böttcher function of a polynomial T. If $z \in \mathbb{C}_+ \cup (\mathbb{R} \setminus J)$, the function $i \log B(z)$ is single-valued. We choose such a branch that $\mathrm{Re}\{i \log B(0)\} = \pi$. We note that if λ increases from 2 to ∞, then $\log |B(0)|$ increases from 0 to ∞. Now we use the functional equation $B(T(z)) = B(z)^2$ and obtain the following statement.

Let $2 < \lambda < \infty$, $\xi = \xi(\lambda) = 1/2 + (\lambda + 1/4)^{1/2} = \max\{x : x \in J\}$, $a = \log |B(0)|$. Then

$$\theta(z) = i \log B(z), \quad z \in \bar{\mathbb{C}}_+; \qquad J = \theta^{-1}\{[0, 2\pi]\}$$

and the balanced measure μ is the preimage of the Lebesgue measure on $[0, 2\pi]$.

Remark. We have obtained the homeomorphism

$$\tilde{\theta}: \quad J \to [0, 2\pi] \Big\backslash \bigcup_{n=0}^{\infty} \bigcup_{q=0}^{2^n - 1} \frac{(2q+1)\pi}{2^n}$$

that yields the semiconjunction of T on J and the function

$$g(x) = \begin{cases} 2x, & x \in [0, \pi] \\ 4\pi - 2x, & x \in [\pi, 2\pi] \end{cases}$$

on $[0, 2\pi]$. Let $\lambda, \lambda' \in (2, \infty)$, and

$$\psi: \quad x + iy \mapsto x + i \frac{\log |B_{\lambda'}(0)|}{\log |B_\lambda(0)|} y$$

be a quasiconformal homeomorphism with a constant distortion. Then $\varphi = \tilde{\theta}_{\lambda'}^{-1} \circ \psi \circ \tilde{\theta}_\lambda$ is the quasiconformal homeomorphism of the complex plane with a constant distortion that gives a conjunction of $T_\lambda |_{J_\lambda}$ and $T_{\lambda'} |_{J_{\lambda'}}$. An existence (but not the explicit form) of such a homeomorphism follows from the profound theorem of Mane, Sad, and Sullivan.[3,4]

3. ITERATIONS OF POLYNOMIALS AND THE DIFFERENCE EQUATION OF THE SCHRÖDINGER TYPE

Let $\mathbb{T}$ be the following RPF operator in the space $C(J)$:

$$(\mathbb{T} g)(x) = \frac{1}{2} \sum_{T(y) = x} g(y)$$

Then the balanced measure μ is an eigenmeasure of the adjoint operator $\mathbb{T}^*$. (Later we will also consider the operator $\mathbb{T}$ in $L^2_{d\mu}$.)

A system of orthonormal polynomials $P_n(z) = P_n(z, \mu)$ was considered in ref. 1. It is easy to deduce from (2.1) that

$$P_n(z^2 - \lambda) = P_{2n}(z), \qquad n \in \mathbb{N} \tag{3.1}$$

In particular, $P_{n^n} = (1/\sqrt{\lambda})\, T_n$. As usual, there exists a three-term recurrence relation

$$[R(k+1)]^{1/2}\, P_{k+1}(z) + [R(k)]^{1/2}\, P_{k-1}(z) = zP_k(z), \qquad k \in \mathbb{Z}_+ \tag{3.2}$$

where $[R(k)]^{1/2} > 0$, $k \in \mathbb{N}$, and $R(0) = 0$.

We denote by H_+ both the Jacobi matrix acting in $l^2(\mathbb{Z}_+)$ associated with the system $\{P_n\}$ and the unitarily equivalent operator of multiplication by the independent variable in the space $L^2_{d\mu}$. Thus, μ is a spectral measure of H_+.

One can check the following operator identity in $L^2_{d\mu}$:

$$\mathbb{T}(H_+^2 - \lambda) = H_+ \mathbb{T} \tag{3.3}$$

By virtue of (3.1) the operator of decimation $\mathbb{D}$ acting in $l^2(\mathbb{Z}_+)$,

$$(\mathbb{D}\psi)(n) = \psi(2n), \qquad n \in \mathbb{Z}_+$$

is unitarily equivalent to the operator $\mathbb{T}$. Thus, Eq. (3.3) yields the renormalization identity

$$\mathbb{D}(H_+^2 - \lambda) = H_+ \mathbb{D}$$

and the following recurrence relations:

$$\begin{gathered} R(0) = 0, \qquad R(2k) + R(2k+1) = \lambda \\ R(2k)\, R(2k-1) = R(k) \end{gathered} \tag{3.4}$$

These relations allow one to prove inductively the estimate[1]

$$\sup_{p, s > 0} |R(p2^n + s) - R(s)| \leqslant \frac{\lambda}{(\lambda - 2)^n} \tag{3.5}$$

If $\lambda > 3$, the latter estimate implies an LP property for the sequence $\{R(k)\}_{k \in \mathbb{Z}_+}$ and hence for the operator H_+. Indeed,

$$R_n(p2^n + k) = R(k), \qquad 0 \leqslant k \leqslant 2^n - 1, \quad p \geqslant 0$$

is a sequence of the period 2^n. It follows from (3.5) that

$$\|R_n - R\|_{l^\infty(\mathbb{Z}_+)} \leqslant \varepsilon(n) \to 0, \qquad n \to \infty$$

The proof of LP was extended[9] to values of $\lambda > 2.192$ and to complex values of λ with a large modulus.

Let us extend the sequence $\{R_n(k)\}$ to negative values of k by periodicity. The extended sequence is fundamental in $l^\infty(\mathbb{Z})$. We denote by $R = R(k)$, $k \in \mathbb{Z}$, its limit when $n \to \infty$, i.e.,

$$R(-k) = \lim_{n \to \infty} R(2^n - k), \qquad k \in \mathbb{N} \tag{3.6}$$

where the limit is uniform with respect to k. We denote by H the extended Jacobi matrix acting in $l^2(\mathbb{Z})$: $h_{k,k+1} = h_{k+1,k} = [R(k)]^{1/2}$, $k \in \mathbb{Z}$, and all other elements of H are equal to zero. Since $R(0) = 0$, the matrix H has a block-diagonal form, i.e., the operator H is decomposable into a direct sum $H = H_+ \oplus H_-$ and we must investigate the operator H_- in $l^2(\mathbb{Z}_+)$.

4. A RESOLVENT FUNCTION AND A SPECTRAL MEASURE OF THE OPERATOR H_-

Let ν be the spectral measure of the operator H_-, and

$$w(z) = w(z; \nu) = \int \frac{d\nu(t)}{z - t}$$

be a resolvent function of H_-. We will express $w(z^2 - \lambda)$ through $w(z)$ and obtain a functional equation for $w(z)$.

Let us introduce the rational functions

$$w_k(z) = \frac{P_{k-1}(z)}{[R(k)]^{1/2} P_k(z)}, \qquad P_k = P_k(\cdot, \mu) \tag{4.1}$$

By using the recurrence relations (3.2), we get

$$w_k(z) = \cfrac{1}{z - \cfrac{R(k-1)}{z - \cfrac{R(k-2)}{z - \cdots - \cfrac{R(1)}{z}}}} \tag{4.2}$$

Due to (3.6), the relation

$$w_{2^n}(z) \to \cfrac{1}{z - \cfrac{R(-1)}{z - \cfrac{R(-2)}{z - \cdots}}} \tag{4.3}$$

holds for large $|z|$. It follows immediately from (4.1) that $w(z) = w(z; \nu)$ is a function analytic outside of a limit set of zeros of the sequence of polynomials $\{P_{2^n}\} = \{(1/\sqrt{\lambda})\, T_n\}$ i.e., outside of the Julia set J. Consequently, $\operatorname{supp}(\nu) \subset J$ and (4.3) is valid in $\mathscr{D}_T(\infty)$.

By virtue of (3.1), (3.4), and (3.2) we have

$$w_{2^n}(z^2 - \lambda) = \frac{1}{R(2^n - 1)} \left[z w_{2^{n+1}}(z) - 1 \right]$$

Passing on to the limit when $n \to \infty$, we deduce

$$w(z^2 - \lambda) = \frac{z w(z) - 1}{R(-1)} \tag{4.4}$$

This is the functional equation for $w(z)$ that we sought. The respective equation for the measure ν is

$$\frac{1}{x^2}\, \mathbb{T}^* \nu = \frac{1}{R(-1)}\, \nu \tag{4.5}$$

where $x \in J$ is an independent variable. We introduce the RPF operator $\mathbb{B} = \frac{1}{2} \mathbb{T} H_-^{-2}$ or, in other words,

$$(\mathbb{B} g)(x) = \sum_{T(y) = x} \frac{g(y)}{|T'(y)|^2} = \frac{1}{4} \sum_{T(y) = x} \frac{g(y)}{y^2}, \qquad g \in C(J)$$

Then, because of (4.5), ν is the eigenmeasure of the adjoint operator $\mathbb{B}^*$:

$$\mathbb{B}^* \nu = \frac{1}{2R(-1)}\, \nu$$

Now we recall the RPF theorem[2,4] for the operator $\mathbb{B}$. Let ρ be a spectral radius of $\mathbb{B}$. Then ρ is a simple eigenvalue of the operators $\mathbb{B}$ and $\mathbb{B}^*$. We denote by $h(x) \in C(J)$ the corresponding eigenfunction for $\mathbb{B}$ and by $\tilde{\nu}$ the corresponding eigenmeasure for $\mathbb{B}^*$. Then $h(x) > 0$, $x \in J$, and $\tilde{\nu}$ is a nonnegative measure. If we choose $h(x)$ in such a way that $\int h\, d\tilde{\nu} = 1$, then for any $g \in C(J)$

$$\lim_{n \to \infty} \left\| \frac{1}{\rho^n}\, \mathbb{B}^n g - \left(\int g\, d\tilde{\nu} \right) h \right\|_{C(J)} = 0 \tag{4.6}$$

It easily follows from (4.6) that $\nu = C\tilde{\nu}$, $C = \text{const} > 0$. The eigenmeasure ν is called the SBR measure. The measure $h\, d\nu$ coincides with a unique Gibbs state on J with respect to the Holder continuous function

$\varphi = -2 \log |T'|$. In particular, the measure ν does not have discrete mass points. It follows from ref. 2 that the balanced measure and the measure $h\,d\nu$ are distinct ergodic measures. Hence they are mutually singular.

5. FURTHER PROPERTIES OF THE OPERATOR

1. It is possible to prove the following equalities similar to (3.1):

$$zP_n(z^2 - \lambda) = P_{2n+1}(z), \qquad n \in \mathbb{Z}_+ \tag{5.1}$$

where $P_n = P_n(\cdot, \nu)$ are orthonormal polynomials with respect to the SBR measure. Let us define another operator of decimation in $l^2(\mathbb{Z}_+)$: $(\tilde{\mathbb{D}}\psi)(n) = \psi(2n+1)$, $n \in \mathbb{Z}_+$. Equalities (5.1) yield the renormalization identity

$$\tilde{\mathbb{D}}(H_-^2 - \lambda) = H_- \tilde{\mathbb{D}}$$

2. One can ask a natural question about the hull[11,12] of the LP sequence R.

Let I_2 be the ring of all entire dyadic numbers with the usual topology. It is not difficult to show that the hull of R is $\{R_\omega\}_{\omega \in I_2} = \mathrm{cl}(\{R_k\}_{k \in \mathbb{Z}})$, where cl denotes closure in $l^\infty(\mathbb{Z})$ and $R_\omega(n) = R(n + \omega)$.

Let $H(\omega)$ be an operator in $l^2(\mathbb{Z})$ generated by the sequence R_ω. If $\omega \in I_2 \backslash \mathbb{Z}$, it is easy to show that $R_\omega(n) \neq 0$, $n \in \mathbb{Z}$, and the operator $H(\omega)$ is not decomposable into a direct sum. It is known[11,12,15] that the spectrum of $H(\omega)$ does not depend on ω; hence, $\sigma(H(\omega)) = \sigma(H) = J$.

3. It is also known[11,12,15] that the integrated density of states (i.d.s.) does not depend on ω. The i.d.s. of the operator H_+ coincides[1] with the distribution function $M(x) = \mu((-\infty, x))$ of the measure μ; therefore the i.d.s. of $H(\omega)$ also coincides with $M(x)$.

Let

$$\mathfrak{M} = \bigcup_{n=0}^{\infty} \bigcup_{q=0}^{2^n - 1} \frac{(2q+1)\,\pi}{2^n}$$

be the frequency module of the sequence R. According to the Johnson–Moser theorem,[15] $M(x) \subset \mathfrak{M}$ if $x \in \mathbb{R} \backslash \sigma(H) = \mathbb{R} \backslash J$. By using the map $\theta(z)$ (see Section 2), one can easily show that in our case $\{M(x) \colon x \in \mathbb{R} \backslash \sigma(H)\} = \mathfrak{M}$, so the gap labeling is complete.

6. THE SBR MEASURE. SOME CALCULATIONS

First we recall that both the SBR measure ν and the operator $\mathbb{B}$ play important roles in the measurable dynamics of an expanding polynomial T

(see ref. 2). Let ρ be a spectral radius of the operator $\mathbb{B}$ in the space $C(J)$. Then by virtue of the RPF theorem the equalities

$$\log \rho = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{B}^n \mathbb{1}$$
$$= \lim_{n \to \infty} \frac{1}{n} \log \sum_{T_n(y) = x} \frac{1}{|T_2'(y)|^2} \tag{6.1}$$

are valid, where the latter limit is uniform with respect to $x \in J$. It follows from (6.1) that $\log 1/\rho$ is equal to an escape rate,[2,10] i.e.,

$$\log \frac{1}{\rho} = \lim_{n \to \infty} \frac{1}{n} \log \frac{1}{\text{area } U(\varepsilon, n)}$$

where $U(\varepsilon) = U(\varepsilon, 0)$ is an ε-neighborhood of the Julia set J; $U(\varepsilon, n) = T_{-n} U(\varepsilon)$ is its full preimage under T_n.

Let us note that

$$\mathbb{B}\mathbb{1} = \frac{1}{2[x - T(0)]}; \qquad \mathbb{B} \frac{1}{x - z} = \frac{1}{2z} \left(\frac{1}{x - T(z)} - \frac{1}{x - T(0)} \right) \tag{6.2}$$

[The operator $\mathbb{B}$ acts on a variable $x \in J$, and $z \in \mathscr{D}_T(\infty)$ is a parameter.] By virtue of (6.2) and the RPF theorem, one can seek the function $h(x) = \lim_{n \to \infty} \rho^{-n} \mathbb{B}^n \mathbb{1}$ in the following form:

$$h(x) = \sum_{k=1}^{\infty} \frac{c_k}{x - T_k(0)}$$

Using again (6.2), we find that

$$h(x) = \text{const} \cdot \sum_{k=1}^{\infty} \frac{(2\rho)^{-k}}{T_1(0) \cdots T_{k-1}(0)} \frac{1}{x - T_k(0)}, \qquad \text{const} > 0$$

where $1/2\rho$ is the zero of the entire function

$$F(t) = 1 + \sum_{k=1}^{\infty} \frac{t^k}{T_1(0) \cdots T_{k-1}(0)\, T_k(0)} \tag{6.3}$$

with the least modulus (such a zero is unique). It should be noted that the series in (6.3) converges very rapidly and that the function $F(t)$ decreases when t grows from zero to infinity. These circumstances allow us to easily find the numerical value of the escape rate in our case (see Fig. 2). It is interesting to observe that the eigenfunction h extends to a meroporphic function in the whole plane.

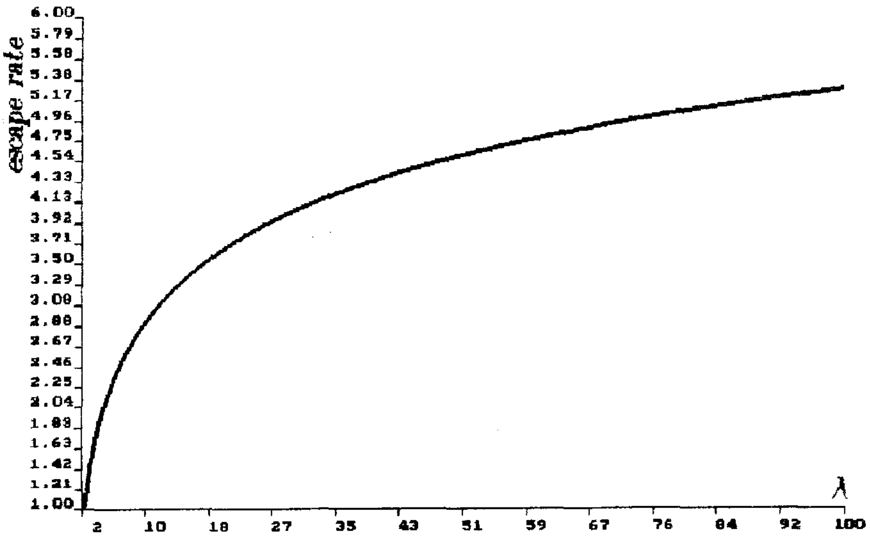

Fig. 2. The escape rate of $T(x)=x^2-\lambda$, $\lambda>2$.

Now we will find an expression for the resolvent function $w(z)=w(z;\nu)$. Let us rewrite Eq. (4.4) in the form

$$w(z)=\frac{1}{z}+\frac{1}{2\rho z}\,w(T(z))$$

Iterating this equality, we obtain the desired series

$$w(z)=\sum_{k=0}^{\infty}\frac{(2\rho)^{-k}}{T_0(z)\,T_1(z)\cdots T_k(z)}$$

This series is well defined on $\mathscr{D}_T(\infty)$ and has the following property: its partial sums

$$\sum_{k=0}^{N}\frac{(2\rho)^{-k}}{T_0(z)\cdots T_k(z)}$$

are $[2^N-2/2^N-1]$ Padé approximants of $w(z)$.

Remark. In ref. 13, devoted to the convergence of $[2^n-2^l-1/2^n-2^l]$ Padé approximants of the Stieltjes transform of the balanced measure μ, Levin introduced an entire function

$$A(t,z)=\sum_{k=0}^{\infty}\frac{t^k}{T_0(z)\cdots T_k(z)}$$

and proved the following remarkable formula:

$$\lambda \frac{A(t, z)}{F(t)} = \sum_{n=1}^{\infty} w_{2^n}(z)\, R(2^n)\, t^n$$

This formula yields the identity

$$\frac{\lambda}{F(t)} = \sum_{n=1}^{\infty} R(2^n)\, t^n$$

ACKNOWLEDGMENTS

We thank A. Eremenko, G. Levin, M. Ljubich, and L. Pastur for valuable discussions and L. Maslov for the computation of the numerical value of the escape rate.

REFERENCES

1. I. Bellissard, D. Bessis, and P. Moussa, *Phys. Rev. Lett.* **49**:701 (1982); M. F. Barnsley, I. Geronimo, and A. Harrington, *Commun. Math. Phys.* **99**:303 (1985): I. Bellissard, in *Trends in the Eighties*, Ph. Blanchard, ed. (Singapore; 1985).
2. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphism* (Springer Verlag, Berlin, 1975).
3. Ph. Blanchard, *Bull. Am. Math. Soc.* **11**:85 (1984).
4. M. Ljubich, *Russ. Math. Surv.* **41**:43 (1986).
5. H. Brolin, *Ark. Mat.* **6**:103 (1965).
6. V. A. Marchenko and I. V. Ostrovskii, *Math. SSSR Sb.* **26**:493 (1975).
7. B. Ja. Levin, in *Complex Analysis*, I. Hersch and A. Huber, eds. (Birkhäuser, Basel, 1988).
8. L. A. Pastur and V. A. Tkachenko, *Sov. Math. Dokl.* **30**:773 (1984).
9. G. A. Baker, Jr., D. Bessis, and P. Moussa, *Physica* **124A**:61 (1984).
10. M. Widom, D. Bensimon, L. Kadanoff, and S. Shenker, *J. Stat. Phys.* **32**:443 (1983).
11. L. A. Pastur, in *Soviet Science Review C Mathematics/Physics*, Vol. 6 (Harwood Academic Publisher, 1987).
12. H. Kunz, R. Livi, and A. Süto, *Commun. Math. Phys.* **122**:643 (1989).
13. G. M. Levin, Polynomial Julia sets and Padé approximations, unpublished (1988).
14. D. Bessis, I. Geronimo, and P. Moussa, *J. Stat. Phys.* **34**:75 (1984).
15. R. Johnson and J. Moser, *Commun. Math. Phys.* **84**:403 (1982); F. Delyon, and B. Souillard, *Commun. Math. Phys.* **89**:415 (1983).

Communicated by L. Pastur.